Clay Mathematics Proceedings
Volume 6, 2006

The Residue Index Theorem of Connes and Moscovici

Nigel Higson

Contents

1. Introduction

Several years ago Alain Connes and Henri Moscovici discovered a quite general "local" index formula in noncommutative geometry [**12**] which, when applied to Dirac-type operators on compact manifolds, amounts to an interesting combination of two quite different approaches to index theory.

Atiyah and Bott noted that the index of an elliptic operator D may be expressed as a complex residue

$$\operatorname{Index}(D) = \operatorname{Res}_{s=0}\big(\Gamma(s)\operatorname{Trace}(\varepsilon(I+\Delta)^{-s})\big),$$

where $\Delta = D^2$ (see [**1**]). Rather surprisingly, the residue may be computed, at least in principle, as the integral of an explicit expression involving the coefficients of D, the metric g, and the derivatives of these functions. However the formulas can be very complicated.

In a different direction, Atiyah and Singer developed the crucial link between index theory and K-theory. They showed, for example, that an elliptic operator D on M determines a class

$$[D] \in K_0(M)$$

2000 *Mathematics Subject Classification.* Primary 58J22, 58J42, 58B34; Secondary 19J56.

in the K-homology of M (see [**2**] for one account of this). As it turned out, this was a major advance: when combined with the Bott periodicity theorem, the construction of $[D]$ leads quite directly to a proof of the index theorem.

When specialized to the case of elliptic operators on manifolds, the index formula of Connes and Moscovici associates to an elliptic operator D on M a cocycle for the group $HCP^*(C^\infty(M))$, the periodic cyclic cohomology of the algebra of smooth functions on M. In this respect the Connes-Moscovici formula calls to mind the construction of Atiyah and Singer, since cyclic cohomology is related to K-homology by a Chern character isomorphism. But the actual formula for the Connes-Moscovici cocycle involves only residues of zeta-type functions associated to D. In this respect it calls to mind the Atiyah-Bott formula.

The proper context for the Connes-Moscovici index formula is the noncommutative geometry of Connes [**7**], and in particular the theory of spectral triples. Connes and Moscovici have developed at length a particular case of the index formula which is relevant to the transverse geometry of foliations [**12, 13**]. This work, which involves elaborate use of Hopf algebras, has attracted considerable attention (see the survey articles [**8**] and [**26**] for overviews). At the same time, other instances of the index formula are beginning to be developed (see for example [**9**], which among other things gives a good account of the meaning of the term "local" in noncommutative geometry).

The original proof of the Connes-Moscovici formula, which is somewhat involved, reduces the local index formula to prior work on the transgression of the Chern character, and is therefore is actually spread over several papers [**12, 11, 10**]. Roughly speaking, the residues of zeta functions which appear in the formula are related by the Mellin transform to invariants attached to the heat semigroup $e^{-t\Delta}$. The heat semigroup figures prominently in the theory of the JLO cocycle in cyclic theory, and so previous work on this subject can now be brought to bear on the local index formula.

The main purpose of these notes is to present, in a self-contained way, a new and perhaps more accessible proof of the local index formula. But for the benefit of those who are just becoming acquainted with Connes' noncommutative geometry, we have also tried to provide some context for the formula by reviewing at the beginning of the notes some antecedent ideas in cyclic and Hochschild cohomology.

As for the proof of the theorem itself, in contrast to the original proof of Connes and Moscovici, we shall work directly with the complex powers Δ^{-z}. Our strategy is to find an elementary quantity $\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_z$ (see Definition 4.12), a sort of multiple zeta function, which is meromorphic in the argument z, and whose residue at $z = -\frac{p}{2}$ is the complicated combination of residues which appears in the Connes-Moscovici cocycle. The proof of the index formula can then be organized in a fairly conceptual way using the new quantities. The main steps are summarized in Theorems 5.5, 5.6, 7.1 and 7.12.

The "elementary quantity" $\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_z$ was obtained by emulating some computations of Quillen [**23**] on the structure of Chern character cocycles in cyclic theory. Quillen constructed a natural "connection form" Θ in a differential graded cochain algebra, along with a "curvature form" $K = d\Theta + \Theta^2$, for which the quantities

$$\Gamma(z) \operatorname{Trace}\big(K^{-z}\big) = \frac{\Gamma(z)}{2\pi i} \operatorname{Trace}\bigg(\int \lambda^{-z} (\lambda - K)^{-1} \, d\lambda\bigg)$$

have components $\langle 1, [D, a^1], \ldots, [D, a^p]\rangle_z$. Taking residues at $z = -\frac{p}{2}$ we get (at least formally)

$$\operatorname{Trace}\big(K^{\frac{p}{2}}\big) = \operatorname{Res}_{z=-\frac{p}{2}} \langle 1, [D, a^1], \ldots, [D, a^p]\rangle_z$$

Now, in the context of vector bundles with curvature form K, the pth component of the Chern character is a constant times $\operatorname{Trace}(K^{\frac{p}{2}})$. As a result, it is natural to guess that our elementary quantities $\langle \cdots \rangle_z$ are related to the Chern character and index theory, after taking residues. All this will be explained in a little more detail at the end of the notes, in Appendix B. Appendix A explains the relation between the Connes-Moscovici cocycle and the JLO cocycle, which was one of the original objects of Quillen's study and which, as we noted above, played an important role in the original approach to the index formula.

A final appendix presents a proof of Connes' Hochschild class formula. This is a straightforward development of the proof of the local index formula presented here. (Connes' Hochschild formula is introduced in Section 3 as motivation for the development of the local index formula.)

Obviously the whole of the present work is strongly influenced by the work of Connes and Moscovici. Moreover, in several places the computations which follow are very similar to ones they have carried out in their own work. I am very grateful to both of them for their encouragement and support. I also thank members of Penn State's Geometric Functional Analysis Seminar, especially Raphaël Ponge, for their advice, and for patiently listening to early versions of this work.

2. The Cyclic Chern Character

In this section we shall establish some notation and terminology related to Fredholm index theory and cyclic cohomology. For obvious reasons we shall follow Connes' approach to cyclic cohomology, which is described for example in his book [**7**, Chapter 3]. Along the way we shall make explicit choices of normalization constants.

2.1. Fredholm Index Problems. A linear operator $T\colon V \to W$ from one vector space to another is *Fredholm* if its kernel and cokernel are finite-dimensional, in which case the *index* of T is defined to be

$$\operatorname{Index}(T) = \dim\ker(T) - \dim\operatorname{coker}(T).$$

The index of a Fredholm operator has some important stability properties, which make it feasible in many circumstances to attempt a computation of the index even if computations of the kernel and cokernel, or even their dimensions, are beyond reach.

First, if $F\colon V \to W$ is any finite-rank operator then $T + F$ is also Fredholm, and moreover $\operatorname{Index}(T) = \operatorname{Index}(T + F)$. Second, if V and W are Hilbert spaces then the set of all bounded Fredholm operators from V to W is an open subset of the set of all bounded operators in the operator norm-topology, and moreover the index function is locally constant. In addition, if $K\colon V \to W$ is any compact operator between Hilbert spaces (which is to say that K is a norm-limit of finite-rank operators), then $T + K$ is Fredholm, and moreover $\operatorname{Index}(T) = \operatorname{Index}(T + K)$. In fact, an important theorem of Atkinson asserts that a bounded linear operator between Hilbert spaces is Fredholm if and only if it is invertible modulo compact operators. See for example [**15**].